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HEAT TRANSFER FOR LAMINAR FLOW OF NON-NEWTONIAN FLUIDS IN ENTRANCE REGION OF A TUBE

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Abstract—This investigation deals with the heat-transfer characteristics of certain non-Newtonian fluids in the entrance region of a circular tube. The particular class of fluids involved is the pseudoplastics, for which the rheological data are best given by

$$\tau = K\left(-\frac{du}{dr}\right)^n \ (0 \leqslant n \leqslant 1).$$

The analysis is restricted to fluids with constant properties in incompressible flow.

The momentum and energy equations were solved analytically for $n = 0$, $n = 0{\cdot}5$, and $n = 1$, the results then compared with known solutions of the two limiting cases: plug flow ($n = 0$) and Newtonian flow ($n = 1$).

The momentum equation was solved by a method analogous to one developed by Goldstein and Atkinson for Newtonian flow. These results are not presented.

Once the velocity distribution is known we can then solve the energy equation by numerical techniques for various boundary conditions, limited in this study to two: constant wall temperature and constant heat flux. Upon solution of the energy equation we then can predict the local and mean Nusselt numbers which can be checked near the entrance by comparison with the modified Leveque and far downstream by comparison to the fully developed velocity solution.

NOMENCLATURE

B, coefficients in finite difference form of energy equation;
c, heat capacity [Btu/lb$_m$-degF];
d, diameter [ft];
h, heat-transfer coefficient [Btu/h-ft^2-degF];
K, rheological constant [lb$_f$ — s^n/ft^2];
k, conductivity [Btu/h-ft-degF];
n, exponent in rheological equation;
q, heat rate [Btu/h];
p_1, dimensionless pressure, $p/g_c \rho u_m^2$;
r, radial dimension [ft];
r_0, radius of tube [ft];
r_1, dimensionless radial distance, r/r_0;
T, temperature [°F];
T_w, wall temperature [°F];
T_i, initial temperature [°F];
u, axial velocity [ft/s];
u_m, mean velocity [ft/s];
u_1, dimensionless velocity, u/u_m;
v, radial velocity [ft/s];
v_1, dimensionless velocity, v/u_m;
x_0, dimensionless distance, z_0/Pr';
z_1, dimensionless axial distance, z/r_0;
z_0, dimensionless axial distance, z_1/Re';
Nu, Nusselt number, hd/k;
Pe, Peclet number, $Re'\,Pr'$;
Pr', modified Prandtl number, $\frac{g_c K c}{k}\left(\frac{r_0}{u_m}\right)^{1-n}$;
Re', modified Reynolds number, $\frac{u_m^{2-n} r_0^n \rho}{g_c K}$
β, dimensionless distance;
θ_1, dimensionless temperature;
θ_m, dimensionless mean temperature;
θ_w, dimensionless wall temperature;
ρ, density [lb$_m$/ft^3];
τ_{rz}, stress [lb$_f$/ft^2].

INTRODUCTION

THE CLASSIC problem of laminar flow of Newtonian fluids in the entrance region of a circular tube has been considered by many investigators, both from the viewpoint of momentum and heat transfer. The most accurate solutions of the momentum equation have been obtained by Atkinson and Goldstein [1], by Punnis [2], and by Langhaar [3]. The first two solutions were achieved by the use of the boundary-layer solution near the entrance and a perturbation solution farther downstream; Langhaar linearized the momentum equation to obtain a single solution for the entire flow regime.

Once the flow pattern is known, the heat transfer can be determined by a numerical solution of the energy equation. Kays [4] used the results of Langhaar to obtain the heat-transfer results for a Prandtl number of 0·7 and for several different wall conditions. These results have been extended to other Prandtl numbers ($Pr = 50$) by Goldberg [5].

This paper presents a similar analysis for non-Newtonian fluids based on the power-law model ($0 \leqslant n \leqslant 1$) and constant properties. The velocity distribution was obtained from a method parallel to that used by Atkinson and Goldstein. In this process the earlier results for Newtonian fluids ($n = 1$) were obtained again, somewhat more extensively, and the heat-transfer results of Kays are extended to Prandtl numbers of 10 and 100. For non-Newtonian fluids the calculations have been carried out for only $n = 0{\cdot}5$, a value representing a significant degree of non-Newtonian behavior. The heat-transfer results for this case are calculated for Prandtl numbers of 1, 10 and 100, and for two wall conditions; constant wall temperature and constant heat flux.

BASIC EQUATIONS

This investigation is directed toward fluids whose rheological data are best given by the empirical power-law, that is $\tau \sim |\mathrm{d}u/\mathrm{d}y|^n$. Here the absolute value is indicated because in the experiments the co-ordinates are so taken that the strain rate is interpreted as a positive value. For more complicated situations, however, in which velocity gradient never changes sign and is always negative, it can be shown [6] that an adequate representation of the shear is

$$\tau = K\left(-\frac{\mathrm{d}u}{\mathrm{d}r}\right)^n \tag{1}$$

This expression is valid in the present case where $0 \leqslant n \leqslant 1$.

In a manner parallel to Atkinson and Goldstein [1, p. 305], we divide the flow regime into two regions; that region near the inlet where the boundary-layer assumptions apply, and that region farther downstream where a perturbation solution is assumed valid. In both regions the radial momentum equations are negligible and the shear gradient in the axial direction is ignored in the equation in the z direction. Then, with the shear relation specified by equation (1), the non-dimensional continuity and momentum equations are:

continuity: $$\frac{\partial(r_1 v_1)}{\partial r_1} + \frac{\partial(r_1 u_1)}{\partial z_1} = 0 \tag{2}$$

momentum: $$u_1 \frac{\partial u_1}{\partial z_1} + v_1 \frac{\partial u_1}{\partial r_1} = -\frac{\mathrm{d}p_1}{\mathrm{d}z_1} - \frac{1}{Re'}\left(-\frac{\partial u_1}{\partial r_1}\right)^{n-1} \frac{\partial}{\partial r_1}\left(r_1 \frac{\partial u_1}{\partial r_1}\right) \tag{3}$$

Here $Re' = (u_m^{2-n} r_0^n \rho / g_c K)$ is the Reynolds number defined for non-Newtonian fluids.

The non-dimensional energy equation is

$$u_1 \frac{\partial \theta_1}{\partial x_0} + Pe\, v_1 \frac{\partial \theta_1}{\partial r_1} = \frac{\partial^2 \theta_1}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial \theta_1}{\partial r_1} \tag{4}$$

Equation (4) ignores the effects of viscous dissipation.

For pipe flow, $v_1 (\partial\theta_1/\partial r_1)$ is small except in the immediate vicinity of the entrance. Kays showed that for Newtonian flow this term amounts to 10 per cent at $(z_1/Re'Pr') = 0{\cdot}01$ for a certain value of r_1. Since the radial convective term appears to be important only in the boundary-layer region, it is taken into account there and neglected in the perturbation region.

The solution of the momentum equation, both in the boundary-layer region and the perturbation region, yields the velocity distribution necessary to solve the energy equations. The details and results of the solution of equation (3) are given in reference [6]. Concurrently with this investigation, Collins and Schowalter [7] have performed a similar analysis of the momentum equation.

Once the velocity distribution is known, we can proceed to evaluate the heat transfer from equation (4). The boundary conditions are:

	constant wall		constant heat flux
$r_1 = 1$	$\theta_1 = 0$	or	$\frac{\partial \theta_1}{\partial r_1} = 1$
$r_1 = 0$	$\frac{\partial \theta_1}{\partial r_1} = 0$	or	$\frac{\partial \theta_1}{\partial r_1} = 0$
$x_0 = 0$	$\theta_1 = 1$	or	$\theta_1 = 0$

where $\theta_1 = (T_w - T/T_w - T_i)$ for constant wall temperature and $\theta_1 = (T - T_i)/(qr_0/k)$ for constant heat flux.

Once the temperature distribution is known for any value of x_0, the local Nusselt number can be evaluated as follows:

constant wall temperature

$$Nu_{z_1} = -\frac{2(\partial \theta_1/\partial r_1)_{r_1=1}}{\theta_m} \qquad (5)$$

where $\theta_m = \int_0^1 2\, r_1\, u_1\, \theta_1\, dr_1$;

constant heat flux $Nu_{z_1} = (2/\theta_w - \theta_m)$ (6)

where $\theta_m = 2x_0$.

NUMERICAL PROCEDURES

Equation (4) was solved numerically by the "Marching Solution", the method used by Kays. Equation (4), in terms of finite differences, is

$$\theta_1 (r_1, x_0 + \Delta x_0) = B_1 \theta_1 (r_1 + \Delta r_1, x_0) + B_2\, \theta_1 (r_1, x_0) + B_3 \theta_1 (r_1 - \Delta r_1, x_0)$$

where $B_1 = \frac{\Delta x_0}{u_1\, \Delta r_1} + \frac{\Delta x_0}{2u_1\, r_1 \Delta r_1} - \frac{\Delta x_0\, Pr'\, v_1}{2u_1\, \Delta r_1}$

$$B_2 = 1 - \frac{2x_0}{u_1\, r_1^2}$$

$$B_3 = \frac{\Delta x_0}{u_1\, \Delta r_1^2} - \frac{\Delta x_0}{2u_1\, r_1\, \Delta r_1} + \frac{\Delta x_0\, Pr'\, v_1}{2u_1\, \Delta r_1}$$

Once Δr_1 is chosen, Δx_0 is not completely arbitrary, since, for convergence of the solution, $B_2 \geqslant 0$. For this problem $\Delta r_1 = 0{\cdot}05$, and $\Delta x_0 = 0{\cdot}0002$. Under these conditions, B_2 is always positive. The temperature gradients at $r_1 = 0$, and $r_1 = 1$ were calculated from the relation

$$\frac{\partial \theta_1}{\partial r_1} = \frac{1}{12\, \Delta r_1} [- 25\theta_1 (j) + 48\theta (j + 1) - 36\theta (j + 2) + 16\theta (j + 3) - 3\theta (j + 4)]. \qquad (7)$$

Kurowski [8] suggests that the best value of the derivative is obtained by including no more than 5 or 6 points. It was decided to use only 5 points. Certainly the value of the derivative and therefore Nu in equation (5) will vary slightly depending upon the number of points chosen.

It was also necessary to specify a modified Prandtl number since the velocities are given in terms of z_0, which is equal to $x_0\, Pr'$. The modified Prandtl number is customarily defined as:

$$Pr' = \frac{Pe}{Re'} = \frac{K_c\, g_c}{k} \left(\frac{r_0}{u_m}\right)^{1-n}.$$

The mean Nusselt number was determined numerically by integrating the local Nusselt number by means of the trapezoidal rule.

RESULTS AND DISCUSSION

Fully developed flow

As a check on the accuracy of the numerical procedures, the energy equation was solved for fully developed flow for $n = 1$, $n = 0{\cdot}5$ and $n = 0$ and the resulting Nusselt numbers calculated and compared to known solutions. The results are given in Table 1. Newtonian flow ($n = 1{\cdot}0$) and plug flow ($n = 0$) are in excellent agreement with existing analytical solutions. These analytical solutions are discussed thoroughly in references [4] and [13] for Newtonian flow and reference [5] for plug flow.

The case of non-Newtonian fluids which follow the power-law model has also been solved previously; by Lyche and Bird [9] for constant wall temperature, and by Bird [10] for constant heat flux. Lyche and Bird present their results in terms of temperature profiles for particular values of x_0 and an average Nusselt number given by

$$\frac{h_a\, d}{k} = \frac{1 - \theta_m}{x_0} \qquad (8)$$

To compare their results with the present investigation, it was necessary to determine $(\partial \theta/\partial r_1)_{r_1=1}$ from their temperature profile to use in equation (5), which probably accounts for the difference in the results shown in Table 1. If, on the other hand, we cast the present results into the form of equation (8) we get good agreement.

Table 1. Nusselt number for fully developed flow

	Constant wall temperature			Constant heat flux		
			$n = 1{\cdot}0$			
x_0	Nu_x	Nu_x*	Nu_m	Nu_x	Nu_x†	Nu_m
0·01	7·44	7·47	12·23	9·37	9·29	14·86
0·02	5·98	6·00	9·43	7·53	7·49	11·60
0·04	4·91	4·92	7·41	6·17	6·15	9·17
0·06	4·43	—	6·49	5·56	5·55	8·06
0·10	4·00	4·00	5·56	4·98	4·97	6·93
0·20	3·71	3·71	4·68	4·52	4·51	5·81
0·30	3·66	3·66	4·35	4·41	4·40	5·36
			$n = 0{\cdot}5$			
x_0	Nu_x	Nu_x‡	Nu_m	Nu_x	Nu_x§	Nu_m
0·01	7·97	—	10·51	10·07	—	15·38
0·02	6·40	—	8·78	8·09	—	12·15
0·04	5·25	—	7·26	6·63	—	9·70
0·08	4·47	—	6·02	5·61	5·56	7·86
0·10	4·29	4·23	5·69	5·37	5·35	7·39
0·20	3·99	3·93	4·89	4·90	4·89	6·23
0·30	3·95	—	4·59	4·79	4·78	5·76
			$n = 0$ (plug flow)			
x_0	Nu_x‖	Nu_x	Nu_m	Nu_x	Nu_x‖	Nu_m
0·01	13·04	—	24·31	20·49	20·38	30·30
0·02	9·89	9·88	17·71	15·35	15·33	23·89
0·04	7·75	—	13·17	11·89	11·88	18·59
0·06	6·89	6·89	11·21	10·46	10·46	16·09
0·10	6·18	6·18	9·33	9·17	9·16	13·54
0·20	5·82	5·82	7·63	8·24	8·24	11·06
0·40	5·78	—	6·71	8·01	8·01	9·57

* Kays [4] (discussion by R. Lipkiss).
† R. Siegel, E. M. Sparrow and T. M. Hallman [13].
‡ Calculated from results in reference [9].
§ Calculated from results in reference [10].
‖ Rohsenow [5], or McKillop [6], for example.

x_0	Present	Lyche and Bird
0·1	4·43	4·42
0·2	3·15	3·15

Using the data presented by Bird the local Nusselt number for constant heat flux was calculated for several values of x_0 (see Table 1). The discrepancy between Bird's results and the current investigation can probably be attributed to the numerical integration necessary to obtain the constants in Bird's equation for the Nusselt number. For large values of x_0 the solutions appear to converge to 4·76, the asymptotic value given by Bird.

Thus, the numerical procedures used in the solution of the fully developed flow case are in excellent agreement with known analytical solutions. The procedures, therefore, should give reliable results for the entrance region. Metzner and Gluck [11] give an excellent discussion on the current status of non-Newtonian heat transfer for fully developed flow and, in particular, present experimental data for the constant wall temperature.

Entrance region

Local values of the Nusselt number for Newtonian flow ($n = 1$), constant wall temperature and a $Pr = 0{\cdot}7$ are given in Table 2 and are shown by curve *c* in Fig. 1, as these were obtained from the present numerical solution for $x_0 > 0{\cdot}005$. At large values of x_0 the Nusselt number converges to the accepted value of 3·66. For values of $x_0 < 0{\cdot}005$, additional considerations are necessary to orient the curves. These begin with the indication of the Pohlhausen (flat-plate) solution, valid very close to the entrance as $x_0 \to 0$, shown by curve *a* in Fig. 1. A connexion to curve *c* is achieved by the Leveque–Lighthill solution, based on an assumed linear velocity profile near the wall and valid when the thermal energy is largely contained near the wall. Thus, the approximation applies well for large Prandtl numbers and, particularly in boundary-layer solutions, reasonable validity has been indicated for Prandtl numbers as low as unity. The solution is specified here in the form given by Acrivos [12], useable for both Newtonian and non-Newtonian fluids.

$$Nu_{z_0} = \frac{2}{\Gamma(4/3)} \frac{Pr'}{9} \frac{\sqrt{\beta}}{[\int_0^{z_0} (\sqrt{\beta})\, dz_0]^{1/3}} \tag{9}$$

Table 4 gives the values of $\sqrt{\beta}$ and $[\int_0^{z_0} (\sqrt{\beta})\, dz_0]^{1/3}$ as they were obtained from the numerical solution of the momentum equation. Equation (9) reduces to the flat-plate solution as $z_0 \to 0$. Also, since $\beta \to \infty$ as $z_0 \to 0$, the value of $\int_0^{z_0} (\sqrt{\beta})\, dz_0$ was determined at $z_0 = 0{\cdot}0001$ from the flat-plate solution.

Equation (9), then, produces the Nusselt numbers shown by curve *b* of Fig. 1. Curves *a*, *b* and *c* present the preferred solution for constant wall temperature and $Pr = 0{\cdot}7$.

Table 2. Local and mean Nusselt numbers ($n = 1{\cdot}0$)

	Constant wall temperature		Constant heat flux	
x_0	Nu_x	Nu_m	Nu_x	Nu_m
		$Pr = 0{\cdot}7$		
0·009	10·04	16·82	13·19	23·78
0·014	8·50	14·04	11·07	19·45
0·024	6·91	11·32	8·93	15·38
0·044	5·53	8·93	7·09	11·93
0·064	4·89	7·75	6·23	10·26
0·084	4·52	7·01	5·72	9·23
0·10	4·27	6·51	5·38	8·52
0·12	4·10	6·13	5·15	7·99
0·15	3·95	5·72	4·90	7·41
0·204	3·80	5·27	4·67	6·76
0·400	3·666	4·49	4·40	5·65
		$Pr = 10$		
0·005	10·13	14·94	12·96	21·15
0·010	7·82	12·03	9·99	16·32
0·020	6·13	9·48	7·78	12·55
0·040	4·95	7·47	6·25	9·73
0·060	4·48	6·55	5·60	8·46
0·080	4·18	5·99	5·24	7·70
0·10	4·01	5·61	5·00	7·18
0·12	3·90	5·33	4·84	6·80
0·15	3·79	5·04	4·67	6·39
0·20	3·71	4·71	4·52	5·94
0·40	3·65	4·20	4·38	5·19
		$Pr = 100$		
0·005	8·55	10·94	11·55	16·62
0·010	7·11	9·37	9·27	13·44
0·020	5·86	7·89	7·50	10·84
0·040	4·86	6·59	6·15	8·78
0·060	4·41	5·93	5·55	7·80
0·080	4·15	5·52	5·20	7·19
0·10	3·99	5·23	4·98	6·77
0·12	3·88	5·01	4·82	6·46
0·15	3·78	4·77	4·66	6·11
0·20	3·70	4·51	4·52	5·73
0·30	3·66	4·23	4·41	5·30

Curve *h* indicates the earlier solution of Kays. This earlier result is somewhat above the present ones because Kays neglected the radial convective

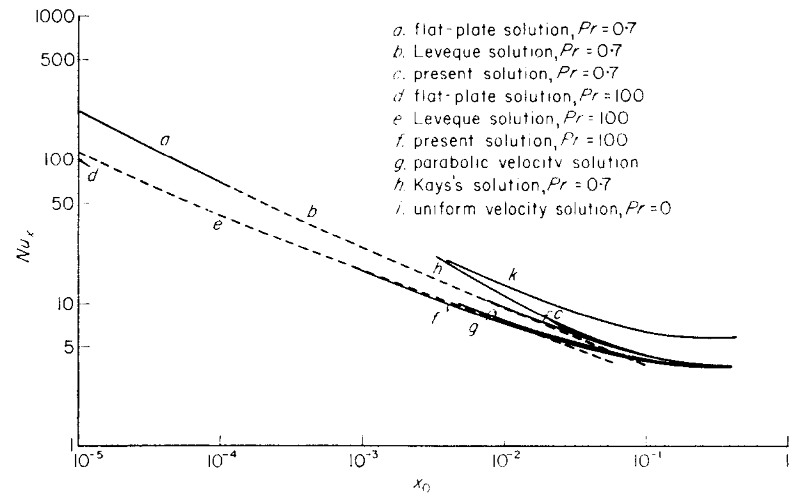

FIG. 1. Local Nusselt number for constant wall temperature and $n = 1$.

component of heat transfer in the boundary-layer region and used the Langhaar velocity distribution. The present solution was re-worked neglecting the radial convective term and the results compare quite favorably with those of Kays.

The effect of the Prandtl number is indicated on Fig. 1 by curves *d*, *e* and *f* ($Pr = 100$), by curves *a*, *b* and *c* ($Pr = 0{\cdot}7$), and by curve *k* ($Pr = 0$). As Kays [4] points out, the uniform velocity represents the case of $Pr = 0$.

Figure 2 is a similar portrayal of local Nusselt numbers for constant heat flux. The Leveque–Lighthill solution is given by equation (9) where the constant is 2·76 instead of 2·0. Again, Kays's solution is high for small values of z_0. At large values of x_0, the solution converges to the accepted value of 4·36.

The values of the local Nusselt number for a non-Newtonian flow ($n = 0{\cdot}5$) with an isothermal wall are shown in Fig. 3 and given in Table 3. Near the entrance, good correspondence exists

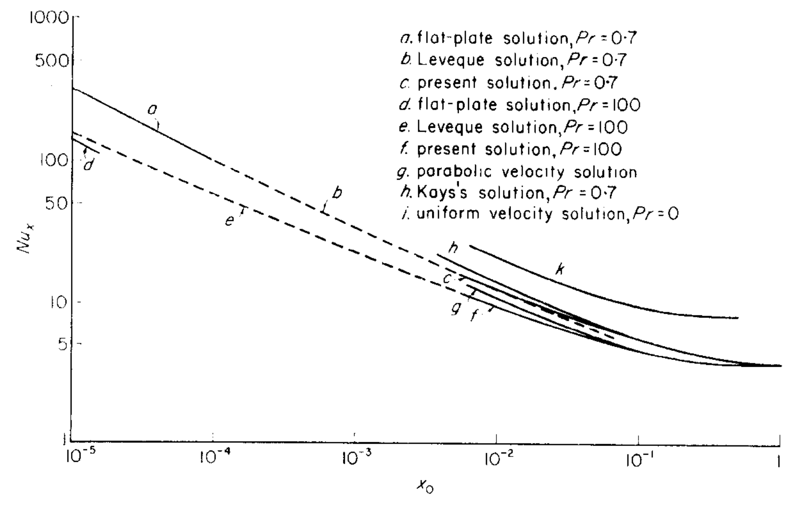

FIG. 2. Local Nusselt number for constant heat flux and $n = 1$.

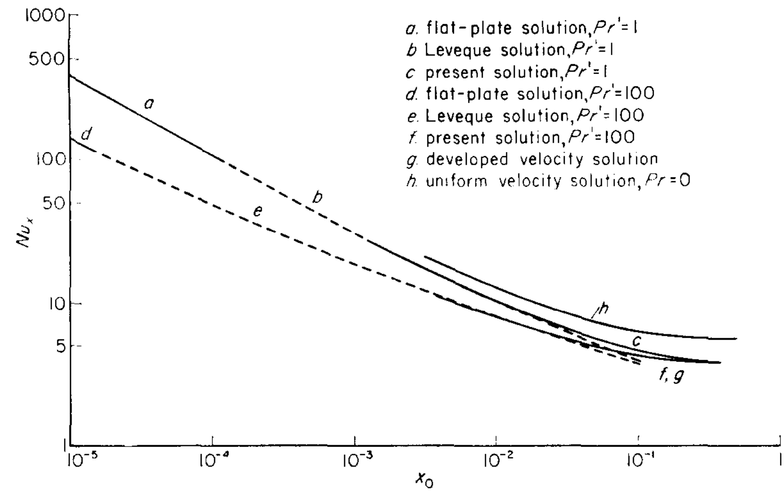

FIG. 3. Local Nusselt number for constant wall temperature and $n = 0.5$.

Table 3. Local and mean Nusselt numbers ($n = 0.5$)

	Constant wall temperature		Constant heat flux	
x_0	Nu_x	Nu_m	Nu_x	Nu_m
		$Pr' = 1.0$		
0.008	10.05	18.27	15.43	27.98
0.014	8.71	14.60	12.07	22.04
0.019	7.95	12.98	10.77	19.29
0.039	6.31	9.95	8.29	14.28
0.059	5.48	8.57	7.15	12.02
0.079	4.95	7.72	6.45	10.70
0.099	4.61	7.13	5.97	9.79
0.12	4.39	6.69	5.64	9.12
0.15	4.18	6.20	5.31	8.39
0.20	4.04	5.67	5.02	7.57
0.30	3.96	5.11	4.82	6.68
		$Pr' = 10$		
0.005	11.32	16.55	14.55	24.52
0.010	8.68	13.31	11.22	18.83
0.020	6.64	10.45	8.52	14.32
0.040	5.32	8.17	6.77	10.94
0.060	4.78	7.13	6.06	9.43
0.080	4.49	6.51	5.66	8.53
		$Pr' = 10$—*continued*		
0.10	4.30	6.08	5.40	7.93
0.12	4.19	5.78	5.23	7.50
0.15	4.08	5.45	5.06	7.03
0.20	4.00	5.10	4.90	6.51
0.30	3.95	4.73	4.79	5.97
		$Pr' = 100$		
0.001	8.01	11.08	10.26	15.59
0.0145	7.02	9.83	8.95	13.50
0.0195	6.41	9.03	8.16	12.23
0.0395	5.26	7.36	6.66	9.72
0.0595	4.75	6.56	6.00	8.57
0.0795	4.47	6.07	5.62	7.87
0.0995	4.29	5.73	5.38	7.39
0.1195	4.18	5.12	5.21	7.04
0.1495	4.08	5.20	5.05	6.66
0.1995	3.99	4.91	4.90	6.23
0.2995	3.95	4.59	4.79	5.76

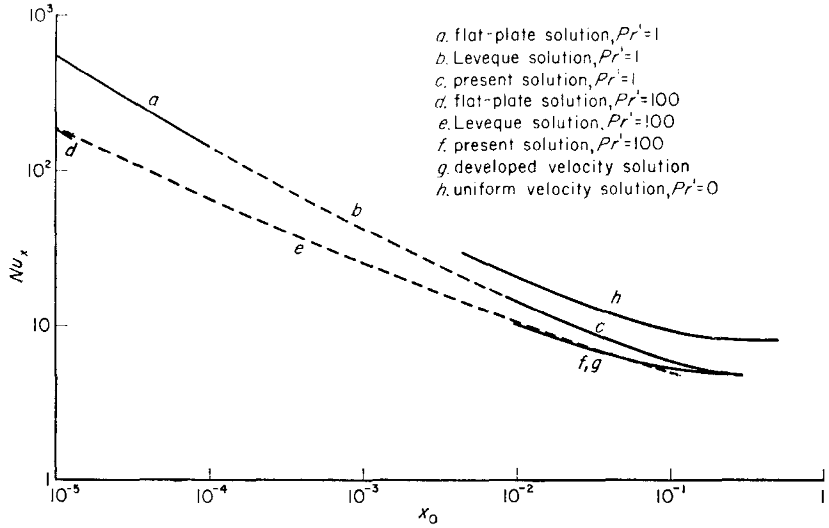

FIG. 4. Local Nusselt number for constant heat flux and $n = 0{\cdot}5$.

Table 4. Functional values of velocity gradient for modified Leveque solution

	$n = 1{\cdot}0$			$n = 0{\cdot}5$	
z_0	$\sqrt{\beta}$	$[\int_0^z (\sqrt{\beta})\,dz_0]^{1/3}$	z_0	$\sqrt{\beta}$	$[\int_0^z (\sqrt{\beta})\,dz_0]^{1/3}$
0·0001	5·78	0·0937	0·0001	13·59	0·137
0·0002	5·26	0·111	0·0002	12·20	0·157
0·0003	4·80	0·123	0·0003	10·56	0·171
0·0004	4·52	0·133	0·0004	9·12	0·182
0·0005	4·32	0·141	0·0005	8·00	0·190
0·0006	4·12	0·148	0·0006	7·20	0·197
0·0007	4·03	0·154	0·0007	6·65	0·203
0·0008	3·93	0·159	0·0008	6·26	0·208
0·0009	3·85	0·164	0·0009	6·02	0·212
0·0010	3·78	0·169	0·001	5·84	0·216
0·0015	3·51	0·188	0·002	5·10	0·250
0·0020	3·36	0·203	0·003	4·62	0·273
0·0025	3·26	0·215	0·004	4·31	0·292
0·0125	2·61	0·340	0·005	4·10	0·308
0·0325	2·33	0·445	0·006	3·95	0·321
0·0525	2·20	0·510	0·007	3·83	0·333
0·0725	2·14	0·561	0·008	3·75	0·344
0·1025	2·08	0·621	0·01	3·65	0·364
0·1225	2·06	0·655	0·03	2·91	0·481
0·1525	2·04	0·700	0·05	2·73	0·552
0·2025	2·03	0·763	0·07	2·59	0·605
0·2525	2·00	0·817	0·10	2·45	0·667
1·0	2·00	1·269	0·14	2·33	0·732
			0·20	2·28	0·809
			0·25	2·25	0·863
			0·35	2·25	0·954
			1·0	2·24	1·33

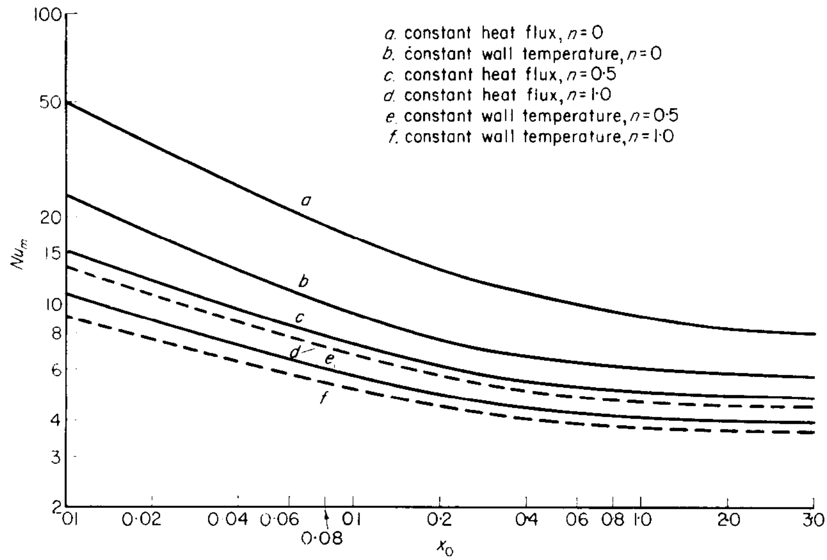

FIG. 5. Mean Nusselt number for $Pr' = 100$.

between the flat-plate, Leveque–Lighthill, and numerical solutions, and for $x_0 \to \infty$ the terminal value is 3·95.

The effect of the Prandtl number is indicated on Fig. 3 by curves d, e and f ($Pr = 100$), by curves a, b and c ($Pr = 1·0$), and curve h ($Pr = 0$).

Figure 4 is a similar portrayal of local Nusselt numbers for constant heat flux. At large values of x_0, the solution appears to converge to 4·76, the asymptotic value.

The effect of n on the Nusselt number is shown in Fig. 5 where the mean Nusselt numbers for various wall conditions and $Pr = 100$ are shown. For $n > 0·5$, the deviation from Newtonian flow is slight, but as $n \to 0$, the effect becomes quite pronounced.

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Résumé—Cette recherche concerne les caractéristiques de transport de chaleur de certains fluides non-newtoniens dans la région d'entrée d'un tube circulaire. La classe particulière de fluides impliquée est celle des pseudo-plastiques, pour laquelle les données rhéologiques sont données le mieux par:

$$\tau = K\left(-\frac{du}{dr}\right)^n (0 \leqslant n \leqslant 1).$$

L'analyse est réduite aux fluides à propriétés constantes en écoulement incompressible.

Les équations de la quantité de mouvement et de l'énergie ont été résolues analytiquement pour $n = 0$, $n = 0{,}5$ et $n = 1$, les résultats étant alors comparés avec des solutions connues des deux cas limites: écoulement en bloc ($n = 0$) et écoulement newtonien ($n = 1$).

L'équation de la quantité de mouvement a été résolue par une méthode analogue à celle exploitée par Goldstein et Atkinson pour l'écoulement newtonien. Ces résultats ne sont pas présentés.

Une fois que la distribution de vitesse est connue, nous pouvons alors résoudre l'équation de l'énergie par des techniques numériques pour différentes conditions aux limites; température de paroi uniforme et flux de chaleur uniforme. A partir de la solution de l'équation de l'énergie, nous pouvons alors prédire les nombres de Nusselt locaux et moyens qui peuvent être vérifiés près de l'entrée par comparaison avec la solution modifiée de Lévêque et plus loin en amont par comparaison avec la solution avec la vitesse entièrement développée.

Zusammenfassung—Die Untersuchung erstreckt sich auf den Wärmeübergang im Einlaufgebiet für Rohre mit Kreisquerschnitt, die von gewissen nichtnewtonschen Flüssigkeiten durchströmt werden. Die untersuchten Flüssigkeiten sind Pseudoplastik-Materialien, deren rheologische Daten am besten durch

$$\tau = K\left(-\frac{du}{dr}\right)^n (0 \leqslant n \leqslant 1)$$

wiedergegeben werden können.

Die Analysis beschränkt sich auf Flüssigkeiten mit konstanten Stoffwerten in inkompressibler Strömung. Bewegungs- und Energiegleichung werden analytisch für $n = 0$, $n = 0{,}5$ und $n = 1$ gelöst und die Ergebnisse mit bekannten Lösungen der beiden Grenzfälle, Kolbenströmung ($n = 0$) und Newton'sche Strömung ($n = 1$) verglichen.

Die Bewegungsgleichung wurde analog einer, von Goldstein und Atkinson für Newton'sche Flüssigkeiten entwickelten Methode gelöst. Diese Ergebnisse sind hier nicht angegeben. Mit bekannter Geschwindigkeitsverteilung lässt sich die Energiegleichung durch numerische Auswertung für verschiedene Randbedingungen lösen, wie es hier für konstante Wandtemperaturen und konstanten Wärmestrom durchgeführt ist. Mit der Lösung der Energiegleichung erhält man die örtlichen und mittleren Nusselt-Zahlen, die, im Einlaufgebiet, durch Vergleich mit der modifizierten Levequelösung, in grösserer Entfernung stromabwärts, durch Vergleich mit der Lösung für ausgebildetes Geschwindigkeitsprofile nachgeprüft werden können.

Аннотация—Проведено исследование теплообменных характеристик некоторых неньютоновских жидкостей во входном участке круглой трубы. Эти жидкости включают особую группу псевдопластиков, реологические данные для которой хорошо описываются уравнением:

$$\tau = K\left(-\frac{du}{dr}\right)^n (0 \leqslant n \leqslant 1)$$

Анализируются только жидкости с постоянными свойствами в несжимаемом потоке.

Уравнения количества движения и энергии решены аналитически для $n = 0$, $n = 0{,}5$ и $n = 1$. Решения затем сравниваются с известными решениями для двух предельных случаев: структурного течения ($n = 0$) и неньютоновской жидкости ($n = 1$).

Уравнение количества движения решено методом, аналогичным методу Гольдстейна и Аткинсона для ньютоновского течения. Эти результаты не приводятся.

Если известно распределение скорости, то уравнение энергии можно решить численным методом для различных граничных условий. В данном исследовании решение дается только для двух случаев: постоянной температуры стенки и постоянного теплового потока. Решив уравнение энергии, можно рассчитать локальное и среднее числа Нуссельта. Результаты для входного участка сравниваются с модифицированными решениями Левека, а для стабилизированного участка дается сравнение с решениями для полностью развитого профиля скорости.